Dream (Physics): φ is a random function on a domain Ω, such that $\forall x \in \Omega$, $\varphi(x)$ is Gaussian, mean zero, and

$$E(\varphi(x)\varphi(y)) = G_\Omega(x,y) - \text{Green function.}$$

$$\left(\forall x \in \Omega \quad h_x(y) := G_\Omega(x,y) \text{ satisfies } \Delta h_x(y) = -\delta_x,\ h_x|_{\partial\Omega} = 0 \text{ q.e.}\right)$$

Note that if we know $E(\varphi(x)\varphi(y))\ \forall x,y \in \Omega$, we know $E(\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n))\ \forall x_1,\ldots,x_n \in \Omega$, by Wick's formula:

$$E(\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)) = \sum_{\substack{\text{perfect pairings} \\ (i_k, j_k) \text{ of} \\ \{1,2,\ldots,n\}}} E(\varphi(x_{i_k})\varphi(x_{j_k})) = \sum G(x_{i_k}, x_{j_k}).$$

In 1D: the dream is realizable!

Take $\Omega = (0,1)$.

$G(x,y) = x(1-y)$ if $0 \le x \le y \le 1$ - Green function

$\varphi(x)$ = Brownian Bridge - Brownian Motion (in x) conditioned so that $\varphi(0) = \varphi(1) = 0$.

Construction: B - standard B.M.

Let $\varphi(x) := (1-x)\, B\left(\frac{x}{1-x}\right)$ - Gaussian, zero mean.

$$E(\varphi(x)\varphi(y)) = E\left((1-x)(1-y)\, B\left(\tfrac{x}{1-x}\right) B\left(\tfrac{y}{1-y}\right)\right) = \qquad E(B(s)B(t)) = \min(s,t)$$

$$(1-x)(1-y)\frac{x}{1-x} = x(1-y).$$

Observe: $\varphi(0) = 0 \quad \varphi(1) = \lim_{x \to 1} (1-x)\, B\left(\frac{x}{1-x}\right) = 0 \quad \left(\text{since } \lim_{t\to\infty} \frac{B(t)}{t} = 0\right).$

Another construction: $\varphi(x) = B(x) - x B(1)$.

What about $\geq 2D$? Can not be a function:

$$E(\varphi(x)\varphi(x)) = G(x,x) = \infty.$$

Gaussian Hilbert Space

Let H be a Hilbert space. Let it be over $\mathbb{R}$, for now.

Def Gaussian Hilbert Space indexed by H:

$\varphi : H \to L^2(\Omega, \mathbb{P})$ – linear, such that

1) $\varphi(h)$ is a zero-mean Gaussian variable
2) $E(\varphi(h)\varphi(g)) = \langle h, g \rangle$.

In particular, φ is an isometry.

Construction: take an orthonormal basis of H (e_n)

Take (α_n) – independent standard Gaussians ($\mathcal{N}(0,1)$ distributed)

Map $\varphi : h = \Sigma a_n e_n \to \Sigma a_n \alpha_n$.

Well-defined, because $\Sigma a_n \alpha_n$ is a Gaussian, with mean zero (1) holds) and variance $\sqrt{\Sigma a_n^2} = \|h\|_H$.

So it is isometry, and, by polarization, $E(\varphi(g)\varphi(h)) = \langle h, g \rangle$.

Case of finite dimension H:

$h = \sum_{k=1}^{n} a_k e_k$, so $\varphi(h) = \sum_{k=1}^{n} a_k \alpha_k = \langle h, \sum_{k=1}^{n} \alpha_k e_k \rangle$.

So $\varphi(h) = \langle \Phi, h \rangle$, where $\Phi = \sum_{k=1}^{n} \alpha_k e_k$ – random element of H.

Distributed with density $e^{-\|x\|_H^2/2}$ in (e_k)-coordinates.

Case of infinite dimensional H:

"$\Phi =$" $\Sigma \alpha_n e_n \notin H$ a.s. Indeed,

$\Sigma \alpha_n^2$ diverges a.s.

Indeed, $\Sigma P(|\alpha_n| > 1) = \Sigma P(|\mathcal{N}(0,1)| > 1) = \infty$.

So, by Borel-Cantelli, $|\alpha_n| > 1$ infinitely often.

So, in this case, Φ becomes a random functional on H, $\varphi(h)$ "$=$" $\langle \Phi, h \rangle$.

notation

Let $\Omega \subset \mathbb{R}^2$ (or $\mathbb{R}^d$) be a domain.

Consider $H = W_0^{1,2}$ – Sobolev space, with

norm $\|f\|_\nabla = \int_\Omega |\nabla f|^2 \, dA$ – Dirichlet norm.

(Formally: closure in L^2 of smooth compactly supported functions with respect to this norm).

By Green's formula,

$$\langle f,g\rangle_\nabla = \int \nabla f \, \nabla g = -\int f \Delta g = -\int \Delta f \, g = -\langle f, \Delta g\rangle$$

(we denote by $\langle\cdot,\cdot\rangle$ the usual L^2 scalar product)

Observe: it is conformally invariant:

if $\varphi: \Omega_1 \to \Omega_2$ - conformal, then

$$\langle f\circ\varphi, g\circ\varphi\rangle_{\nabla,\Omega_1} = \int_{\Omega_1} \nabla f(\varphi)\cdot|\varphi'| \, \nabla g(\varphi) \, |\varphi'| = \int_{\Omega_2} \nabla f \, \nabla g = \langle f,g\rangle_{\nabla,\Omega_1}$$

Notation: $H(\Omega)$

Def. Gaussian Free Field on Ω with zero boundary values is Gaussian Hilbert space indexed by $H(\Omega)$, i.e. random functional $h \to \langle \Phi, h\rangle_\nabla$, $\Phi = \sum \alpha_i e_i$ ($\{e_i\}$ - ONB of $H(\Omega)$)

A.s., $\Phi \notin H(\Omega)$.

Since H is conformally invariant, so is Φ:

if $\varphi: \Omega_1 \to \Omega_2$, then $\Phi_{\Omega_2}\circ\varphi = \Phi_{\Omega_1}$.

Example: $\Omega = (0,\pi)^2$ - square.

Take the basis $e_{m,n} = \dfrac{\sin mx \cos ny}{\sqrt{m^2+n^2}}$.

$$\Phi = \sum \frac{\alpha_{m,n}}{\sqrt{m^2+n^2}} \sin mx \cos ny.$$

$\dfrac{\alpha_{m,n}}{\sqrt{m^2+n^2}}$ is Gaussian, with varianc $\dfrac{1}{m^2+n^2}$.

Note that $\sum \dfrac{1}{m^2+n^2} = \infty$, so $\sum \dfrac{\alpha_{m,n}}{\sqrt{m^2+n^2}} \sin mx \cos ny$ - diverges a.s.

Not a function!

But if we add any power of $\sqrt{m^2+n^2}$, then
$\sum \dfrac{\alpha^2_{m,n}}{(m^2+n^2)} \left(\sqrt{m^2+n^2}\right)^{-2\varepsilon}$ converges for any $\varepsilon > 0$

(Since $\sum \dfrac{1}{(m^2+n^2)^{1+\varepsilon}} < \infty$)

So antiderivative of Φ of any order is an a.s. L^2 function!

So $\Phi \in W^{s,2}$ for any $s < 0$.

By conformal invariance, in any simply connected Ω,
$\Phi \in W^{s,2}_{loc}$ a.s.

$\Phi \in W^{s,2}_{loc}$ a.s.

GFF acting on test functions:

Let $\rho \in C_0^\infty(\Omega)$ be a test function.

Want to compute $\langle \Phi, \rho \rangle$.

Notation: $G\rho(x) = \int_\Omega G_\Omega(x,y)\rho(y)\,dy$ – Green potential.

$\Delta G\rho = \int_\Omega \Delta G(x,y)\rho(y)dy = -\rho(x)$

$\Rightarrow G\rho \in H(\Omega)$, with norm

$\|G\rho\|^2_{H(\Omega)} = \langle G\rho, G\rho\rangle_\nabla = -\langle G\rho, \Delta G\rho\rangle = \langle G\rho, \rho\rangle = \iint G(x,y)\rho(x)\rho(y)$

Green energy.

So, since $\langle \Phi, h\rangle_\nabla = -\langle \Phi, \Delta h\rangle$, we have

$\langle \Phi, G\rho\rangle_\nabla = -\langle \Phi, \Delta G\rho\rangle = \langle \Phi, \rho\rangle$.

So we can think of Φ as a random distribution.

For every ρ, $\langle \Phi, \rho\rangle = \langle \Phi, G\rho\rangle_\nabla$ is normal, zero mean, variance is $\|G\rho\|^2_{H(\Omega)} = \iint G(x,y)\rho(x)\rho(y)$

Green potential

Also, the covariance

$E(\langle \Phi, \rho_1\rangle\langle \Phi, \rho_2\rangle) = \langle G\rho_1, G\rho_2\rangle_\nabla = \iint G(x,y)\rho_1(x)\rho_2(y)$.

What about $E(\varphi(x)\varphi(y))$?

Let us compute it in the sense of distributions:

if ρ_1, ρ_2 – test functions, then

$E(\langle \Phi, \rho_1\rangle\langle \Phi, \rho_2\rangle) = E(\int \Phi(x)\rho_1(x)dx \int \varphi(y)\rho_2(y)dy) =$

Imagine Φ is a "function".

$\iint E(\Phi(x)\Phi(y))\,\rho_1(x)\rho_2(y)$.

So, by above, $G(x,y) = E(\Phi(x)\varphi(y))$ in the sense of distributions.

But Φ is much more than a distribution!

If μ is a measure with finite Green energy

$\left(\left|\iint G(x,y)\,d\mu(x)\,d\mu(y)\right| < \infty \right)$ then

$\langle \Phi, \mu\rangle = $ "$\int \Phi(x)\,d\mu(x)$" $= \langle \Phi, G\mu\rangle_\nabla$ – normal with variance $\iint G(x,y)\, d\mu(x)\, d\mu(y)$

So we can average φ over intervals, any sets of positive capacity. But cannot evaluate at a point!

Restriction to a subdomain.

Let $\Omega' \subset \Omega$ - subdomain.

Then $H(\Omega) = H(\Omega') \oplus \left(H(\Omega')\right)^{\perp}$

What is $H(\Omega')^{\perp}$?

Let $f \in H(\Omega')^{\perp}$. Then $\forall\, g \in H(\Omega')$ we have

$0 = \langle f, g\rangle_{\nabla} = -\langle \Delta f, g\rangle$. So $\Delta f \equiv 0$ in Ω' as distribution, so f is harmonic in Ω' (by Poincare Lemma)

Distributional Laplacian.

$H(\Omega')^{\perp} = \{ f \in H(\Omega) : f \text{ is harmonic in } \Omega' \}$.

So for $h \in H(\Omega)$, $\quad Pr_{H(\Omega')^{\perp}} h = \begin{cases} h(x), & x \notin \Omega', \\ \tilde{h}(x), & x \in \Omega', \end{cases}$

where $\tilde{h}$ - solution of Dirichlet problem with Boundary Data h. $\tilde{h}$ - harmonic in Ω'.

$Pr_{H(\Omega')} h = h - \tilde{h}$

Thus we can write

$\varphi_{\Omega} = \varphi_{\Omega'} + \varphi_{\Omega'}^{\perp}$, with

$\langle \varphi_{\Omega'}, h\rangle = \langle \varphi_{\Omega}, h - \tilde{h}\rangle$

$\langle \varphi_{\Omega'}^{\perp}, h\rangle = \langle \varphi_{\Omega}, \tilde{h}\rangle$

$\varphi_{\Omega'}$ and $\varphi_{\Omega'}^{\perp}$ are independent

(this is the general property: if $H = H_1 \oplus H_2$, then φ_{H_1} is independent from φ_{H_2} - just look at the basis!)

Moreover, if $\rho \in C_0(\Omega')$, then

$\langle \varphi^{\perp}_{\Omega'}, \rho \rangle_\nabla = 0$, so

$\langle \Delta \varphi^{\perp}_{\Omega'}, \rho \rangle = 0 \Rightarrow$ $\varphi^{\perp}_{\Omega'}$ is a (random) harmonic function in Ω'.

Boundary conditions.

φ have "zero boundary values".

Let $h \in C(\partial\Omega)$, $\tilde{h}$ – harmonic extension (solution of Dirichlet problem).

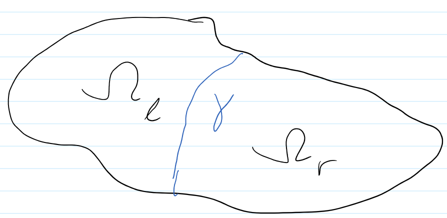
Def Gaussian Free Field in Ω with boundary data h is given by $\varphi^h_\Omega := \varphi_\Omega + \tilde{h}$.

Domain Markov Property.

Given φ_Ω in $\Omega \setminus \Omega'$, $\varphi_\Omega|_{\Omega\setminus\Omega'} = \psi$

φ_Ω is the sum of the $\tilde{\psi} := \begin{cases} \psi \text{ in } \Omega\setminus\Omega' \\ \tilde{\psi} \text{ - harmonic extension of } \psi \text{ in } \Omega' \end{cases}$

and an independent (from $\tilde{\psi}$) $\varphi_{\Omega'}$.



Let γ be a crosscut in Ω, separating ~~it~~ into domains Ω_ℓ and Ω_r.

For $f \in C(\gamma)$, let f_ℓ, f_r be harmonic extensions of f to Ω_ℓ, Ω_r with zero boundary data on $\partial\Omega$

$\tilde{f}(x) := \begin{cases} f_\ell(x), & x \in \Omega_\ell \\ f_r(x) & x \in \Omega_r \end{cases}$

$$\tilde{f}(x) := \begin{cases} f_\ell(x), & x \in \Omega_\ell \\ f(x), & x \in \gamma \\ f_r(x), & x \in \Omega_r \end{cases}$$

Let $\|f\|_N^2 := \int_\Omega |\nabla \tilde{f}|^2$.

$N := \mathrm{Clos}_{\|\cdot\|_N} \{ f \in C(\gamma) \}$

If γ is smooth, then by Green formula

$$\|f\|_N^2 = \|\tilde{f}\|_{H(\Omega)} = \int_\gamma f(\partial_n f_r - \partial_n f_\ell)\, d\ell$$

$f \to \partial_n f_r - \partial_n f_\ell$ is called Dirichlet-to-Neumann operator

If $f \in H(\Omega)$, we can write

$f = \tilde{f}|_\gamma + (f - f_r)|_{\Omega_r} + (f - f_\ell)_{\Omega_\ell}$, all are orthogonal.

So $H(\Omega) = H(\Omega_\ell) \oplus N \oplus H(\Omega_r)$.

So $\varphi_\Omega = \underbrace{\varphi_{\Omega_\ell} + \varphi_{\Omega_r} + \varphi_\gamma}_{\text{independent.}}$

As before, φ_γ is harmonic function in $\Omega_\ell \cup \Omega_r$

φ_γ is the restriction of φ_Ω to γ.

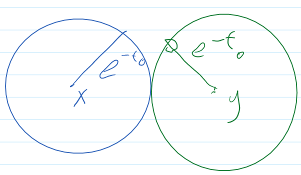
Circle Averages.

Let ℓ_t be normalized length on the circle $C_t^x := \{ z : |z - x| = e^{-t} \}$.

$B_x(t) := \langle \varphi, \ell_t \rangle$ - the average of GFF over C_t^x. $D_t^x := \{ z : |z - x| < e^{-t} \}$.

Well-defined, since ℓ_t has finite energy.

Observe: let $t_0 := \log \frac{2}{|x-y|}$ (i.e. $2e^{-t_0} = |x-y|$).



Then for $t > t_0$, $s > t_0$,

$B_t^x - B_{t_0}^x$ depends only on $\varphi_{D_{t_0}^x}$ - by Markov.

$B_s^y - B_{t_0}^y$ —"— —— on $\varphi_{D_{t_0}^y}$ —"—

$B^y_s - B^y_{t_0}$ — " — on $\varphi_{D^y_{t_0}}$ — " —

So, since $D^x_{t_0} \cap D^y_{t_0} = \emptyset$, these increments are independent!

For fixed x, if $t > s$ then $B^x_t - B^x_s$ is independent of B^x_s (again, domain Markov Property: $B^x_t - B^x_s$ is a function of $\varphi_{D^x_s}$, B^x_s - function of $\varphi^{\perp}_{D^x_s}$).

Also $\mathrm{Var}(B^x_t - B^x_s) = E((B^x_t - B^x_s)^2) \overset{\text{Markov}}{=} \|\ell_t\|^2_{H(D^x_s)} = \iint G_s(x,y)\, d\ell_t(x)\, d\ell_t(y) \ominus$

$G_s(x,y)$ - Green function in D^x_s,

$$G_s(x,y) = \log|x-y| - \log(|x| \cdot |x - y^*|) + s.$$

where y^* - reflection of y wrt C_s.

So direct computation gives

$\ominus$ $\sqrt{2\pi}\,(t-s)$.

So $\frac{1}{\sqrt{2\pi}} B^x_t$ is Brownian Motion.

Let $\Lambda = (V, E)$ - graph, $\partial\Lambda \subset V$ - boundary.
$w: E \to \mathbb{R}_+$ - weight function.

Define $H(\Lambda, \partial\Lambda) = \{\varphi \in \mathbb{R}^V : \varphi|_{\partial\Lambda} = 0\}$.
Inner product on $H(\Lambda, \partial\Lambda)$: $\langle \varphi_1, \varphi_2 \rangle_\nabla = \sum_{e=\langle x,y\rangle} w(e)(\varphi_1(y) - \varphi_1(x))(\varphi_2(y) - \varphi_2(x))$

The corresponding Gaussian Hilbert space Φ on $H(\Lambda, \partial\Lambda)$ defines a random element of $H(\Lambda, \partial\Lambda)$.
The density (wrt Lebesque measure on $\mathbb{R}^V$) is
$\frac{1}{Z} e^{-\|\varphi\|_\nabla^2/2}$ (Z - normalizing factor).

Φ is called <u>discrete GFF</u> or <u>harmonic crystal</u>.

<u>Discrete Laplacian</u>: $\Delta\varphi(x) := \sum_{e=\langle x,y\rangle} w(e)(\varphi(x) - \varphi(y))$.

<u>Observe:</u>

$$\langle \varphi_1, \varphi_2 \rangle_\nabla = \sum_{e=\langle x,y\rangle} w(e)(\varphi_1(y) - \varphi_1(x))(\varphi_2(y) - \varphi_2(x)) =$$

$$\frac{1}{2}\sum_{x\in V} \varphi_1(x) \sum_{e=\langle x,y\rangle} 2(\varphi_2(y) - \varphi_2(x)) = \langle \varphi_1, -\Delta\varphi_2 \rangle \text{ - usual scalar product.}$$

Let X_n^x be weighted random walk started at some $x \in V$,
i.e. $X_0^x = x$, $P(X_{n+1}^x = y \mid X_n^x = z) = \frac{w(\langle z,y\rangle)}{\sum_{z\in e} w(e)}$.

$N^x := \min\{n : X_n^x \in \partial\Lambda\}$ - hitting time.

Define discrete Green function G_x as

$G_x(y) = E\left(\sum_n 1\{X_n^x = y, n < N^x\}\right)$ - expected number of visits to y.

<u>Observe:</u> $G_x(y) = G_y(x)$

Indeed: $G_x(y) = \sum_{\substack{\gamma\text{-pass in} \\ \Lambda\setminus\partial\Lambda \text{ from } x \text{ to } y}} P(\gamma) = G_y(x)$, since each pass from x to y has corresponding pass from y to x.

Observe: 1) $G_x|_{\partial\Lambda} = 0$

2) for $y \neq x$, $\Delta G_x(y) = 0$.

indeed, $G_x(y) = G_y(x) = E\left(\sum_{n\geq 1} 1_{\{X_n^y = x,\, n < N^y\}}\right) =$

$E\left(\sum_{n\geq 1} 1_{\{X_n^{(y)} = x,\, n < N^x\}} \mid X_1^y = z\right) P(X_1^y = z) \overset{\text{Markov}}{=}$

$$\sum_{z\sim y} \frac{w_{(y,z)}}{\sum_{y\in e} w_e} G_z(x) = \sum_{z\sim y} \frac{w_{(y,z)}}{\sum_{y\in e} w_e} G_x(z).$$

3) for x: $\Delta G_x(x) = -1$ — the same reasoning!

Discrete Green Potential: $G\rho(x) = \sum G_x(y)\, \rho(y)$

Observe: $\Delta G\rho(x) = \sum \Delta_x G_x(y)\, \rho(y) = \sum \underbrace{\Delta_x G_y(x)}_{(=0 \text{ if } y\neq x)} \rho(y) = -\rho(x)$

So we have, for any $\varphi, \rho \in H(\Lambda, \partial\Lambda)$,

$\langle \varphi, \rho \rangle = \langle \varphi, G\rho \rangle_\nabla$.

and

$\langle \varphi, G_x \rangle_\nabla = \langle \varphi, \delta_x \rangle = \varphi(x)$

Apply it to discrete GFF Φ:

$$E(\Phi(x)\Phi(y)) = E(\langle \Phi, G_x \rangle, \langle \Phi, G_y \rangle) \overset{\text{GFF isometry.}}{=} \langle G_x, G_y \rangle_\nabla = G_x(y).$$

As for continuous case, there is <u>Domain Markov Property</u>.

Let $\Lambda_1 \subset \Lambda \setminus \partial\Lambda$. $\partial\Lambda_1 = \Lambda \setminus \Lambda_1$.

Φ be dGFF on $(\Lambda, \partial\Lambda)$.

Φ^1 — dGFF on $(\Lambda_1, \partial\Lambda_1)$

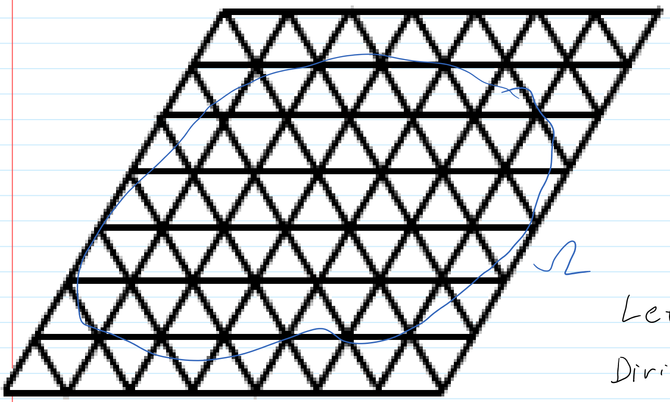
Then $\{\Phi \mid \Phi|_{\partial\Lambda_1} = \psi\} = \Phi_1 + \tilde{\psi}$, where

$\tilde{\psi}$ - discrete harmonic extension of ψ to Λ_1.

$(\Psi(X) := E(\psi(X^x_{N^1_x})),\ N^1_x = \min\{n: X^x_n \notin \Lambda_1\})$.

Discrete approximation.

Let T_n be standard triangular lattice of size $\frac{1}{n}$,



$\Lambda_n(\Omega)$ - graph with vertices $T_n \cap D$, $\partial\Lambda_n$ - vertices with an edge leading outside. Put an equal weight of $\frac{1}{\sqrt{3}}$ on each edge.

Let $H_n = H_{(\Lambda_n(\Omega), \partial\Lambda_n)}$ corresponding Dirichlet space.

Embedding in $H(\Omega)$ - extend by linearity in every triangle.

Denote the extension by the same φ.

For a triangle $[xyz]$ in T_n,

area of $T_n = \frac{\sqrt{3}}{4n^2}$, $\underbrace{|\nabla\varphi|^2}_{\text{continuous}} = \frac{2}{3}n^2((\varphi(x)-\varphi(y))^2 + (\varphi(y)-\varphi(z))^2 + (\varphi(z)-\varphi(x))^2)$

So $\int_{[x,y,z]} |\nabla\varphi|^2 = \frac{\sqrt{3}}{6}((\varphi(x)-\varphi(y))^2 + (\varphi(y)-\varphi(z))^2 + (\varphi(z)-\varphi(x))^2)$

and $\|\varphi\|^2_{H(\Omega)} = \frac{1}{\sqrt{3}}\sum_{x\sim y}(\varphi(x)-\varphi(y))^2 = \|\varphi\|^2_{H_n(\Omega)}$.

So the dGFF Φ_n is the orthogonal projection of GFF Φ to $H_n(\Omega)$, i.e.

$\forall g \in H(\Omega)$, $\langle\Phi_n, g\rangle = \langle\Phi, Pr_{H_n} g\rangle$.

Easy to see: $\bigcup H_n(\Omega)$ is dense in $H(\Omega)$.

Thus $\forall g \in H(\Omega)$, $\|g - Pr_{H_n} g\|_\nabla \to 0$.

So $\forall g \in H(\Omega)$, $\langle\Phi_n, g\rangle \to \langle\Phi, g\rangle$.

Other lattices: can be done the same way, by selecting appropriate weights.

Connection 1: level lines of dGFF.

Works on any lattice, but will work with triangular.
dGFF will be defined on faces of hexagonal (dual to triangular) lattice.

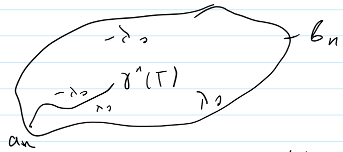
Take $\Omega \subset \mathbb{C}$ – simply connected, Ω_n – lattice approximation.
φ_n – dGFF on Ω_n with boundary values $\lambda > 0$ on the arc from a_n to b_n, $-\lambda$ – on the arc from b_n to a_n.

Assign "+" – to all hexagons with $\varphi_n > 0$,
"−" – to all hexagons with $\varphi_n < 0$ (a.s. $\varphi_n \neq 0 \; \forall$ face).

Then $\exists$ unique interface between "+" and "−" from a_n to b_n. γ^n.

Theorem (Schramm – Scheffield).
γ^n converges to SLE_4 when $\lambda_0 = \sqrt{\frac{\pi}{8}}$
(for other λ – get other versions of SLE)

The key step:
Height-gap lemma. Let $T < \infty$, consider
F_T – solution of discrete Dirichlet problem on $\Omega \setminus \gamma^n_{[0,T]}$ with boundary values λ_0 on $(\gamma^n(T), b_n)$,
$-\lambda_0$ on $(b_n, \gamma^n(T))$



$-\lambda_0$ on $(b_n, \gamma^n(T))$

H_T - expected value of dGFF with boundary conditions:

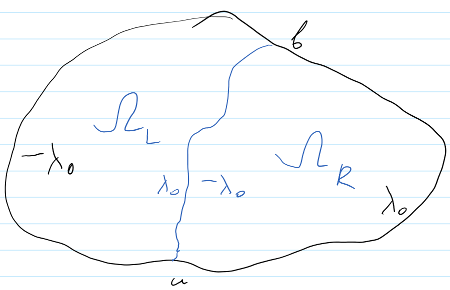
1) λ_0 on (a_n, b_n)
2) $-\lambda_0$ on (b_n, a_n)
3) H on hexagons next to $\gamma^n(T)$.

Then $\forall\, v_0 \in \Omega$, $H_T(v_0) - F_T(v_0) \xrightarrow{P} 0$ as $n \to \infty$.

Remark. If $H_T(v_0) = F_T(v_0)$ - harmonic explorer

Connection 2. Conditioning on SLE_κ.

$\kappa = 4$



Let φ - GFF on Ω with boundary conditions λ_0 on (a,b), $-\lambda_0$ on (b,a).

Let γ - SLE_4 from a to b, Ω_L, Ω_R - random.
Let φ_L be GFF in Ω_L with $(-\lambda_0, \lambda_0)$ b/c
φ_R - GFF in Ω_R with $(-\lambda_0, \lambda_0)$ b/c

Then $\varphi_L \mathbb{1}_{\Omega_L} + \varphi_R \mathbb{1}_{\Omega_R} \overset{law}{=} \varphi$.

Other SLE_κ - more complicated boundary conditions.

$$h_{\lambda,\mu}(w) := \begin{cases} \sqrt{\frac{\pi}{2}}\,\lambda + \mu(\pi - \mathrm{wind}(b,w)), & w \in (a,b) \\ -\sqrt{\frac{\pi}{2}}\,\lambda + \mu(-\pi + \mathrm{wind}(b,w)), & w \in (b,a) \end{cases}$$

As before, wind is the winding.
Formally defined only for smooth curves, but can be extended to arbitrary using conformal ~~maps~~.

If $\lambda = \sqrt{\frac{2}{\pi\kappa}}$, $\mu = \lambda\left(1 - \frac{\kappa}{4}\right)$, then the same observation as for SLE_4 works for any SLE_κ.